MATHEMATICS

INPUT-OUTPUT MARKOV PROCESSES [1])

BY

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1. *Introduction*

In this paper we discuss two stochastic processes which we call input-output Markov processes and in which we use infinite matrices. The results illustrate applications of theorems that have been proved in a previous paper [6].

The first process is suggested by a problem discussed by CHANDRASEKHAR [1]. The second is a simple modification of the first. We can describe each process as consisting of a "reservoir" into which objects flow and out of which objects depart. In each process we assume that the input probability distribution is independent of the number of objects in the reservoir.

We give some rules for the existence of stationary distributions, and also illustrate how to derive them whenever they exist.

2. *Notation, definitions, preliminary results*

1. An infinite stochastic matrix is an array $A \equiv [a_{i,j}]$, $(i,j=0, 1, 2, \ldots)$, with

$$\text{(i)} \quad a_{i,j} \geqslant 0 \text{ for all } i,j,$$

and

$$\text{(ii)} \quad \sum_0^\infty{}_j\, a_{i,j} = 1 \text{ for all } i.$$

2. If for every i we can find a j_i' such that $a_{i,j}=0$ for $j>j_i'$ and $a_{i,j_i'}\neq 0$, he matrix A is called row-finite. Given a row-finite matrix A, we define another matrix $'A \equiv [\alpha_{i,j}]$ by

$$\alpha_{i,j} = \begin{cases} a_{i,j_i'-j} \text{ for } 0 \leqslant j \leqslant j_i', \\ 0 \text{ for } j > j_i', \end{cases}$$

and call *$'A$ the row-reverse of A.*

3. We shall use the following stochastic matrices: *an input transition matrix $\mathscr{B}$* having equal rows $b_0, b_1, b_2, \ldots,$

$$(2.1) \qquad \mathscr{B} \equiv \begin{bmatrix} b_0 & b_1 & b_2 & . & . & , \\ b_0 & b_1 & b_2 & . & & . \\ . & . & . & . & . & . \end{bmatrix},$$

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[1]) These results form part of a thesis for the London Ph.D. degree (1955).

an output transition matrix A which is a lower semi-matrix, i.e.

(2.2) $$a_{i,i} \neq 0, \text{ and } a_{i,j} = 0 \text{ for } j > i.$$

From the input transition matrix $\mathscr{B}$ we form the upper semi-matrix B,

(2.3) $$B \equiv \begin{bmatrix} b_0 & b_1 & b_2 & . & . \\ 0 & b_0 & b_1 & . & . \\ 0 & 0 & b_0 & . & . \\ . & & . & & . \end{bmatrix}.$$

The elements of these matrices will have the following interpretations:

b_k is the probability that k objects flow into the reservoir in a time interval, later to be specified,

$a_{n,k}$ is the probability that if there are n objects in the reservoir, k depart during a time interval, later to be specified.

4. The definitions of an irreducible matrix, of an ergodic matrix, and of a stationary distribution are taken from FELLER [3], and can also be found in our paper [6].

5. We define $\mu = \sum_0^\infty {}_k\, kb_k$ as the *mean input*, when finite. The quantities $\lambda_n = \sum_0^n {}_k\, ka_{n,k}$, $n = 0, 1, 2, \ldots$, will be called *mean outputs*.

6. We say "λ is boundedly greater than μ" if there exists an integer n_0 such that $\inf_{n > n_0} (\lambda_n - \mu) > 0$.

7. The following theorems are of use in the present work:

Theorem 2.I, FOSTER [4]. *An irreducible system represented by the stochastic matrix* $P \equiv [p_{i,j}]$ *is ergodic if for some* $\varepsilon > 0$ *there exists a non-negative solution* $\{y_i\}$ *of the inequations*

$$\sum_0^\infty {}_j\, p_{i,j}\, y_j \leqslant y_i - \varepsilon \quad \textit{for } i > i_0,$$

such that $\sum_0^\infty {}_j\, p_{i,j}\, y_j < \infty$ *for* $i \leqslant i_0$.

Theorem 2.II, [6], p. 238. *The* $(C, 1)$*-limits* Π^c *and* Π^e *of the sequences* $\{C^n\}$ *and* $\{E^n\}$ *are related by* $\Pi^c = A\Pi^e B$ *and* $\Pi^e = B\Pi^c A$, *where* $C = AB$ *and* $E = BA$.

Theorem 2.III, [6], p. 240. *In order that the chain represented by* $C = AB$ *be irreducible, it is necessary that* B *has no zero columns.*

Theorem 2.IV, [6], p. 241. *A necessary and sufficient condition that, if one of* C *or* E *is ergodic the other is also ergodic, is that neither* A *nor* B *have zero columns.*

3. *Simple input-output process*

We consider a reservoir in which there are n objects at time t_0 $(n = 0, 1, 2, \ldots)$ and out of which k objects $(k = 0, 1, 2, \ldots, n)$ may depart during the time interval $t_0 < t \leqslant t_0 + \frac{1}{2}$ with probability $a_{n,k}$. We also suppose that during the time interval $t_0 + \frac{1}{2} < t \leqslant t_0 + 1$ any number m ($m = 0, 1, 2, \ldots$) of objects may flow in with probability b_m.

Let C be the transition matrix of the system after both output and input actions have taken place during the unit time interval, i.e. $c_{n,j}$ is the probability that the system starting at t_0 with n objects in the reservoir ends at $t_0 + 1$ with j objects. We have $c_{n,j} = \sum_{n-j}^{n} {}^{k}\, P(k$ objects depart out of $n)$. $P(j+k-n$ objects flow in), i.e.

$$c_{n,j} = \sum_{n-j}^{n} {}^{k}\, a_{n,k}\, b_{j+k-n} = \sum_{n-j}^{n} {}^{k}\, \alpha_{n,n-k}\, b_{j+k-n} = \sum_{0}^{j} {}^{p}\, \alpha_{n,p}\, b_{j-p},$$

where $'A \equiv [\alpha_{n,k}]$ is the row-reverse of the lower semi-matrix A. Therefore C is given either by the matrix convolution $C = 'A * \mathscr{B}$, i.e. $c_{n,k} = \sum_{0}^{k} {}^{j}\, \alpha_{n,j}\, b_{k-j}$ ([7], p. 160) or equivalently by the matrix product (see formulae (2.1), (2.2), (2.3) and [6], § 4)

$$C = 'A \cdot B. \tag{3.1}$$

4. *Compound input-output process*

If at time t_0 there are n objects in the reservoir, let $e_{n,j}$ be the probability that j objects remain at time $t_0 + 1$ on the assumption that the number n of objects in the reservoir may change in the following ways: k objects may flow in during the time interval $t_0 < t \leqslant t_0 + \frac{1}{2}$ with probability b_k ($k = 0, 1, \ldots$) where b_k does not depend on n; $n+k-j$ objects may flow out during the time interval $t_0 + \frac{1}{2} < t \leqslant t_0 + 1$, with probability $a_{n+k,n+k-j}$ if there are $n+k$ objects present at time $t_0 + \frac{1}{2}$ ($k = 0, 1, \ldots; j = 0, 1, \ldots, n+k$). Then $e_{n,j} = \sum_{0}^{\infty} {}^{k}\, P(k$ objects flow in). $P(n+k-j$ objects depart out of $n+k)$, i.e.

$$e_{n,j} = \sum_{0}^{\infty} {}^{k}\, b_k\, a_{n+k,n+k-j} = \sum_{0}^{\infty} {}^{k}\, b_k\, \alpha_{n+k,j} = \sum_{0}^{\infty} {}^{r}\, b_{r-n}\, \alpha_{r,j}.$$

Hence the transition matrix E of this system is given by the matrix product

$$E = B \cdot 'A. \tag{4.1}$$

From (3.1) and (4.1) we have the following conclusion:

If A is the output transition matrix, $\mathscr{B}$ the input transition matrix, $'A$ the row-reverse of A, and B the upper semi-matrix formed from $\mathscr{B}$, then the transition matrix of the simple input-output process is the matrix product $C = 'A \cdot B$, whereas that of the compound input-output process is the matrix product $E = B \cdot 'A$.

5. *The main results*

Having seen that the transition matrices of the simple and compound input-output processes are the matrix products $'A \cdot B$ and $B \cdot 'A$, we can apply theorems 2.II–2.IV when the stationary distribution of one of the processes is known. We proceed to study the existence and derivation of such distributions.

Theorem 5.I. *If the mean output is boundedly greater than the mean input, and if $C = 'A \cdot B$ is irreducible, then the simple input-output process is ergodic.*

Proof. We have $\lambda_n \geqslant \mu + \varepsilon$ for $n > n_0$, where $\varepsilon > 0$, therefore μ is finite and $\lambda_n + \mu < \infty$ for $n \leqslant n_0$. This together with the identity $\sum_0^n{}_k\, k a_{n,k} = n - \sum_0^n{}_k\, k \alpha_{n,k}$ gives $\sum_0^\infty{}_k\, k\alpha_{n,k} + \sum_0^\infty{}_k\, k b_k \leqslant n - \varepsilon$ for $n > n_0$, and the sum is finite for $n \leqslant n_0$. Since $C = 'A * \mathscr{B}$, it follows from [6], § 4 that $\{c_{n,k}\}$ is the distribution of the *sum of two random variables* with distributions $\{\alpha_{n,k}\}$ and $\{b_k\}$ respectively, therefore taking mean values we have that $\sum_0^\infty{}_k\, k(\alpha_{n,k} + b_k) = \sum_0^\infty{}_k\, k c_{n,k}$, and $\sum_0^\infty{}_k\, k c_{n,k} \leqslant n - \varepsilon$ for $n > n_0$ and is finite for $n \leqslant n_0$. Thus theorem 2.I holds, i.e. C is ergodic.

Remark. A simple case when C is irreducible is given whenever $a_{n,k} > 0$ for $k \leqslant n$, and $b_k > 0$ at least for $k = 0$ and $k = 1$.

Theorem 5.II. *Given that $C = 'A \cdot B$ is irreducible, that $'A$ has no zero columns, and that λ is boundedly greater than μ, then $E = B \cdot 'A$ is ergodic.*

Proof: Theorem 5.I shows that C is ergodic. Since $'A$ has no zero columns, theorems 2.III and 2.IV show that E is ergodic.

6. *Examples and derivation of stationary distributions*

The following problem considered by Chandrasekhar [2] is quoted in Feller [3], p. 344: A process with states $E_0, E_1, E_2, \ldots$ has transition probabilities

$$(6.1) \quad c_{n,k} = e^{-\lambda} \sum_0^n{}_\nu \binom{n}{\nu} p^\nu q^{n-\nu} \lambda^{k-\nu}/(k-\nu)!, \quad 0 < q < 1 \text{ and } p = 1 - q,$$

where the terms in the sum should be replaced by zero when $\nu > k$. Feller states: "This chain occurs in statistical mechanics and can be interpreted as follows: The state of the system is defined by the number of particles in a certain volume of space. During each time interval of unit length each particle has probability q to leave the volume, and the particles are statistically independent. Moreover new particles may enter the volume, and the probability of r entrants is given by the Poisson expression $e^{-\lambda}\lambda^r/r!$." This process can be regarded as a simple input-output process. We can show as an example that this process is ergodic, using our results in section 5. The output transition matrix is given by

$$a_{n,k} = \begin{cases} \binom{n}{k} q^k p^{n-k} & \text{for } 0 \leqslant k \leqslant n, \\ 0 & \text{for } k > n, \end{cases}$$

and the input transition matrix is given by

$$b_k = e^{-\lambda}\,\lambda^k/k! \text{ for } k = 0, 1, 2, \ldots.$$

The transition matrix C is then that given in (6.1), which is the convolution $'A * \mathscr{B}$ i.e. the matrix product $'A \cdot B$. The mean output $\lambda_n = \sum_0^n{}_k\, ka_{n,k} = nq$, and the mean input $\mu = \sum_0^\infty{}_k\, kb_k = \lambda$. We can take n_0 the smallest integer greater than λ/q, so that the mean output is boundedly greater than the mean input, hence by theorem 5.I C is ergodic.

We now consider the *compound* input-output process, with transition matrix $E = B \cdot {}'A$. The stationary distribution of C, given in FELLER [3], p. 345,

$$x_k = e^{-\lambda/q}\,(\lambda/q)^k/k!, \quad k = 0, 1, 2, \ldots$$

can be used to obtain the stationary distribution of the compound input-output process. In this case particles that enter the volume may leave during the same unit interval in which they enter. Since $'A$ has no zero columns, the simple input-output process is ergodic, hence, by theorem 5.II so is the compound input-output process.

Denote the stationary distribution of C by $x \equiv \{x_k\}$, and define

$$y = x'A. \tag{6.2}$$

Then

$$y = x'A = (xC)'A = (x'AB)'A = (x'A)\cdot(B'A) = yE$$

so y is the stationary distribution of E, and we have

$$x = x'AB = yB. \tag{6.3}$$

Therefore

$$y_k = \sum_k^\infty{}_i\, x_i\, \alpha_{i,k} = \sum_k^\infty{}_i\, e^{-\lambda/q}\,\frac{(\lambda/q)^i}{i!}\binom{i}{i-k} q^{i-k}\,p^k = e^{-(\lambda p/q)}\,(\lambda p/q)^k/k!.$$

Therefore while the stationary distribution of the simple input-output process is the Poisson distribution with parameter λ/q, that of the compound process is the Poisson distribution with parameter $\lambda p/q$.

Formulae (6.2) and (6.3) enabled us to calculate the stationary distribution of one process when that of the other was known. If neither is known, the following method may be of use in determining both distributions. The matrix B is an upper semi-matrix, and if $b_0 > 0$, it has a unique upper semi-left- and right-hand reciprocal B^{-1}. We have $x \cdot {}'A \cdot B = x$, and multiplying on the right by B^{-1}, we obtain the equation $x \cdot {}'A = x \cdot B^{-1}$, which, when B^{-1} can be found, may be solved for the stationary distribution x. From (6.2) then y is given by $y = x \cdot {}'A$.

As an example let A be the Cesàro matrix of order $r > 0$, [2], p. 68) and let

$$b_k = \beta(1-\beta)^k,$$

where $0 < \beta < 1$.

Therefore $'A \equiv [\alpha_{n,k}]$ is given by

$$\alpha_{n,k} = \begin{cases} \binom{k+r-1}{k} \Big/ \binom{r+n}{n} & \text{for } 0 \leqslant k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

The mean output

$$\lambda_n = \sum_0^n {}_k\, k \binom{n-k+r-1}{n-k} \Big/ \binom{r+n}{n} = n/(r+1),$$

and the mean input $\mu = \sum_0^\infty {}_k\, kb_k = (1-\beta)/\beta$, hence the mean output is boundedly greater than the mean input. Therefore by theorem 5.I C is ergodic, and so is E by theorem 5.II. Having thus established the existence of the stationary distributions of both C and E, we proceed to derive them.

We have

$$B^{-1} = \frac{1}{\beta} \begin{bmatrix} 1, & \beta-1, & 0\ , & 0\ , & & . \\ 0, & 1\ , & \beta-1, & 0\ , & , & . \\ 0, & 0\ , & 1\ , & \beta-1, & & . \\ . & . & . & . & . & . \end{bmatrix},$$

and therefore

$$\sum_k^\infty {}_i\, x_i\, \alpha_{i,k} = (\beta-1)\, x_{k-1}/\beta + x_k/\beta \ \text{ for } k \geqslant 1,$$

and

$$\sum_0^\infty {}_i\, x_i\, \alpha_{i,0} = x_0/\beta\,.$$

Solving these equations successively we obtain

$$x_k = \binom{r+k}{k} (1-\beta)^k\, x_0\,;$$

and using the fact that $\sum_0^\infty {}_k\, x_k = 1$, we find that

$$x_k = \binom{r+k}{k} (1-\beta)^k\, \beta^{r+1}$$

is the stationary distribution of $C = 'A \cdot B$. Using $y = x \cdot 'A$, we finally obtain:

$$y_k = \binom{r+k-1}{k} (1-\beta)^k\, \beta^r,$$

the stationary distribution of $E = B \cdot 'A$.

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